

The Dirichlet problem for the prescribed Ricci curvature equation on cohomogeneity one manifolds

Artem Pulemotov^{*†}

a.pulemotov@uq.edu.au

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Abstract

Let M be a domain enclosed between two principal orbits on a cohomogeneity one manifold M_1 . Suppose T and R are symmetric positive-definite invariant $(0,2)$ -tensor fields on M and ∂M , respectively. The paper studies the prescribed Ricci curvature equation $\text{Ric}(\mathcal{G}) = T$ for a Riemannian metric \mathcal{G} on M subject to the boundary condition $\mathcal{G}_{\partial M} = R$ (the notation $\mathcal{G}_{\partial M}$ here stands for the metric induced by \mathcal{G} on ∂M). Imposing a standard assumption on M_1 , we prove local solvability and describe a set of requirements on T and R that guarantee global solvability.

1 Introduction

Suppose M is a manifold of dimension 3 or higher (possibly with boundary) and T is a $(0,2)$ -tensor field on M . The present paper investigates the existence of solutions to the prescribed Ricci curvature equation

$$\text{Ric}(\mathcal{G}) = T, \tag{1.1}$$

where the unknown \mathcal{G} is a Riemannian metric on M . This equation relates to a number of fundamental questions in geometric analysis and mathematical physics. For instance, D. DeTurck's work on (1.1) underlay his subsequent discovery of the famous DeTurck trick for the Ricci flow. There is kinship between (1.1) and the Einstein equation from general relativity. Mathematicians have been studying (1.1) since at least the early 1980's. We invite the reader to see [6, 4] for the history of the subject. The list of recent references not mentioned in [6, 4] includes but is not limited to [13, 12, 24].

Several results regarding local solvability of (1.1) are available in the literature. To give an example, suppose o is a point in the interior of M and the tensor field T is nondegenerate at o . It is well-known that (1.1) then has a solution in a neighbourhood of o . There are many different ways to prove this fact. We refer to [22, 6, 23] for more information. Global solvability (i.e., solvability on all of M) of equation (1.1) has been studied rather extensively in the case where $\partial M = \emptyset$. For instance, the work [14] assumes that M is equal to \mathbb{R}^d or an open ball in \mathbb{R}^d . This work provides a sufficient condition for the existence of a solution to (1.1) in the class of metrics on M invariant under the standard action of the special orthogonal group $\text{SO}(d)$. It also describes situations where such a solution cannot be constructed. The reader may consult [18] for cognate material.

The solvability of boundary-value problems for equation (1.1) is, by and large, an unexplored topic. The author of the present paper made progress on this topic in [26]. The main theorems of [26] concern the solvability of Dirichlet- and Neumann-type problems for (1.1) in a neighbourhood of a boundary point on M . These theorems require rather strong assumptions on the tensor field T . Roughly speaking, they demand that T be represented by a nondegenerate diagonal matrix whose components depend on at most one coordinate. Few results concerning global solvability of boundary-value problems for (1.1) previously appeared in the literature. One may be able to obtain such results in the case where M is a closed ball

^{*}School of Mathematics and Physics, The University of Queensland, St Lucia, QLD 4072, Australia

[†]Department of Mathematics, The University of Chicago, 5734 South University Ave, Chicago, IL 60637-1514, USA

in \mathbb{R}^d (possibly with a neighbourhood of the center removed) employing the methods from [14]. However, adapting the arguments in [14] to more general situations seems problematic.

We mentioned above that D. DeTurck's study of (1.1) underlay the discovery of the DeTurck trick. In a similar fashion, new knowledge about the solvability of boundary-value problems for (1.1) may help answer questions about boundary-value problems for the Ricci flow and the Einstein equation. Such questions were studied in [27, 1, 15, 5, 2, 16, 25] and other works. A large number still remain open.

The present paper investigates a Dirichlet-type problem for equation (1.1) on M . Suppose M_1 is a connected manifold with $\partial M_1 = \emptyset$. It will be convenient for us to assume that M is the closure of a domain in M_1 . We concentrate on the case where M_1 is a cohomogeneity one manifold and M is contained between two principal orbits. Our goal is to prove local solvability and provide a sufficient condition for global solvability of a Dirichlet-type boundary-value problem for (1.1) on M . In order to describe the results, we need to explain our setup more rigorously. Consider a compact Lie group G acting on M_1 . Suppose the orbit space M_1/G is one-dimensional. It is then customary to call M_1 a cohomogeneity one manifold. Such manifolds enjoy numerous applications in geometry and mathematical physics: They helped produce important examples of Einstein metrics (see, e.g., [7, 8]). They were effectively used in [10] to study Ricci solitons. For more applications, consult the references of [21]. We assume G acts on M_1 with the principal orbit type G/K whose isotropy representation splits into pairwise inequivalent irreducible summands. This assumption is quite standard; it previously occurred in, e.g., [9, 10]. Section 5.2 discusses an alternative to it. In what follows, we suppose M is the closure of a domain on M_1 contained between the principal G -orbits Γ^0 and Γ^1 . The boundary of M is then equal to the union $\Gamma^0 \cup \Gamma^1$.

The main results of the present paper are Theorems 3.1, 3.2, and 3.4. Let us briefly describe them. Assume the tensor field T is G -invariant and positive-definite (actually, we can replace the latter assumption with a substantially lighter nondegeneracy-type assumption). Consider a principal G -orbit Γ^τ lying in the interior of M . Theorem 3.1 establishes the existence of a G -invariant solution to (1.1) on a neighbourhood of Γ^τ . We explain in Proposition 3.3 what data is needed to determine such a solution uniquely. The reader will find related material in [3, 26].

Given a Riemannian metric \mathcal{G} on M , suppose $\mathcal{G}_{\partial M}$ is the metric induced by \mathcal{G} on ∂M . Consider a symmetric G -invariant tensor field R on ∂M . We supplement (1.1) with the Dirichlet-type boundary condition

$$\mathcal{G}_{\partial M} = R. \tag{1.2}$$

Theorem 3.2 asserts the existence of a G -invariant solution to problem (1.1)–(1.2) in a neighbourhood of ∂M . Again, one can understand from Proposition 3.3 what data determines such a solution uniquely. Note that the boundary condition (1.2) played a major part in the arguments of [3, 26]. It also came up in discussions of the Ricci flow and Einstein metrics; see, e.g., [20, 1, 3].

As we indicated above, few results concerning global solvability of boundary-value problems for (1.1) previously appeared in the literature. We obtain one such result in the present paper. More precisely, our Theorem 3.4 provides a sufficient condition for global solvability of problem (1.1)–(1.2) in the class of G -invariant metrics. This condition consists in a series of inequalities for the tensor fields T and R . Its intuitive meaning is explained in Remark 3.5.

We end the paper with an example. To be more specific, we apply our theorems in the case where M is a solid torus less a neighbourhood of the core circle. Note that our discussion yields alternative proofs of some of the results from [26].

2 The Setup

Suppose M is a smooth manifold with boundary ∂M . Let T be a symmetric $(0, 2)$ -tensor field on M . This paper investigates the equation

$$\text{Ric}(\mathcal{G}) = T \tag{2.1}$$

for a Riemannian metric \mathcal{G} on M . The notation $\text{Ric}(\mathcal{G})$ in the left-hand side stands for the Ricci curvature of \mathcal{G} .

Let $\mathcal{G}_{\partial M}$ be the Riemannian metric induced by \mathcal{G} on ∂M . Given a symmetric $(0,2)$ -tensor field R on ∂M , we supplement (2.1) with the boundary condition

$$\mathcal{G}_{\partial M} = R \quad (2.2)$$

Our intention is to study the solvability of (2.1)–(2.2) in the case where M is a portion of a cohomogeneity one manifold contained between two principal orbits. We will now explain our setup in a more detailed fashion.

2.1 The manifold M

Consider a compact Lie group G acting on a smooth connected manifold M_1 without boundary. Suppose the orbit space M_1/G is one-dimensional. For the sake of convenience, we will assume that M_1/G is homeomorphic to \mathbb{R} . It is easy, however, to state analogues of our results in the situations where this assumption does not hold. We explain this further in Section 5.1.

Fix a point $o \in M_1$. Let K be the isotropy group of o . Consider a diffeomorphism

$$\Phi : \mathbb{R} \times (G/K) \rightarrow M_1$$

such that the map $\Phi(s, \cdot)$ is G -equivariant for every $s \in \mathbb{R}$. We choose two real numbers $\sigma' < \sigma''$ and define

$$M = \Phi([\sigma', \sigma''] \times G/K).$$

Clearly, M is a manifold with boundary. In what follows, we assume the dimension of M is greater than or equal to 3. It is easy to see that ∂M coincides with the union $\Gamma^0 \cup \Gamma^1$, where

$$\Gamma^0 = \Phi(\{\sigma'\} \times G/K), \quad \Gamma^1 = \Phi(\{\sigma''\} \times G/K).$$

Our goal in this paper is to study the prescribed Ricci curvature equation (2.1) on M . In particular, we will discuss the existence of its solutions subject to (2.2).

Let \mathfrak{g} be the Lie algebra of G . Pick an $\text{Ad}(G)$ -invariant scalar product Q on \mathfrak{g} . Suppose \mathfrak{k} is the Lie algebra of K and \mathfrak{p} is the orthogonal complement of \mathfrak{k} in \mathfrak{g} with respect to Q . We standardly identify \mathfrak{p} with the tangent space of G/K at K . The isotropy representation of G/K then yields the structure of a K -module on \mathfrak{p} . The following requirement will be imposed throughout Sections 2, 3, and 4.

Hypothesis 2.1. *The K -module \mathfrak{p} appears as an orthogonal sum*

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_n \quad (2.3)$$

of pairwise non-isomorphic irreducible K -modules $\mathfrak{p}_1, \dots, \mathfrak{p}_n$.

Hypothesis 2.1 is rather standard. It came up in several papers such as [9, 10]. We discuss an alternative to this hypothesis in Section 5.2.

Remark 2.2. Assume G is the special orthogonal group $\text{SO}(d)$ and M_1 equals \mathbb{R}^d less a closed ball around the origin. One may then be able to study problem (2.1)–(2.2) on M with the methods of [14]; see also [18]. We will not explore this in the present paper.

2.2 The tensor fields T and R

Consider a symmetric G -invariant $(0,2)$ -tensor field T on M . We assume T is positive-definite; however, as we explain in Section 5.3, this assumption can be lightened. It is possible to construct a diffeomorphism

$$\Psi : [0, 1] \times (G/K) \rightarrow M$$

such that $\Psi(t, \cdot)$ is G -equivariant for each $t \in [0, 1]$ and the equality

$$\Psi^*T = \sigma^2 dt \otimes dt + \hat{T}_t, \quad t \in [0, 1], \quad (2.4)$$

holds true; see, e.g., [10]. The letter σ here denotes a positive real number. The tensor field \hat{T}_t , defined for each $t \in [0, 1]$, is a G -invariant $(0, 2)$ -tensor field on G/K . Note that \hat{T}_t is fully determined by how it acts on \mathfrak{p} . Furthermore, there exist smooth functions $\hat{\phi}_1, \dots, \hat{\phi}_n$ from $[0, 1]$ to $(0, \infty)$ such that

$$\hat{T}_t(X, Y) = \hat{\phi}_1(t) Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \dots + \hat{\phi}_n(t) Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \quad X, Y \in \mathfrak{p}. \quad (2.5)$$

The notation $\text{pr}_{\mathfrak{p}_k} X$ and $\text{pr}_{\mathfrak{p}_k} Y$ refers to the orthogonal projections of X and Y onto \mathfrak{p}_k for $k = 1, \dots, n$.

Let us consider, along with T , a symmetric positive-definite G -invariant $(0, 2)$ -tensor field R on ∂M . We will write R^0 and R^1 for its restrictions to Γ^0 and Γ^1 , respectively. Without loss of generality, assume that

$$\Gamma^0 = \Psi(\{0\} \times G/K), \quad \Gamma^1 = \Psi(\{1\} \times G/K).$$

The tensor field R is fully determined by how $(\Psi(0, \cdot))^* R^0$ and $(\Psi(1, \cdot))^* R^1$ act on \mathfrak{p} . There exist positive numbers a_1, \dots, a_n and b_1, \dots, b_n satisfying the equalities

$$\begin{aligned} ((\Psi(0, \cdot))^* R^0)(X, Y) &= a_1^2 Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \dots + a_n^2 Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \\ ((\Psi(1, \cdot))^* R^1)(X, Y) &= b_1^2 Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \dots + b_n^2 Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \end{aligned} \quad X, Y \in \mathfrak{p}. \quad (2.6)$$

3 The results

In this section, we formulate the main results of the paper. Recall that Hypothesis 2.1 is imposed. Equalities (2.4), (2.5), and (2.6) hold for the tensor fields T and R .

3.1 Local solvability

To begin with, we discuss local solvability of problem (2.1)–(2.2). Given $\tau \in [0, 1]$, let Γ^τ stand for the G -orbit $\Psi(\{\tau\} \times G/K)$ on M . This is consistent with the notation Γ^0 and Γ^1 introduced above. The first question we address is whether equation (2.1) can be solved in a neighbourhood of Γ^τ when Γ^τ lies in the interior of M (i.e., when $\tau \in (0, 1)$). Under our current assumptions, the answer turns out to be positive.

Theorem 3.1. *For each $\tau \in (0, 1)$, there exist a neighbourhood \mathcal{X}^τ of the set Γ^τ in M and a G -invariant Riemannian metric \mathcal{G}^τ on M such that $\text{Ric}(\mathcal{G}^\tau) = T$ on \mathcal{X}^τ .*

We are also able to prove that problem (2.1)–(2.2) has a solution near $\partial M = \Gamma^0 \cup \Gamma^1$. More precisely, the following result holds.

Theorem 3.2. *There exist a neighbourhood $\mathcal{X}^{0,1}$ of ∂M and a G -invariant Riemannian metric $\mathcal{G}^{0,1}$ on M such that $\text{Ric}(\mathcal{G}^{0,1}) = T$ on $\mathcal{X}^{0,1}$ and $\mathcal{G}_{\partial M}^{0,1} = R$.*

The proofs of Theorems 3.1 and 3.2 will rely on Proposition 3.3 appearing below. In order to formulate this proposition, we need more notation. Given $\tau \in [0, 1]$ and $\kappa > 0$, set

$$\mathcal{X}_\kappa^\tau = \Psi(((\tau - \kappa, \tau + \kappa) \cap [0, 1]) \times G/K).$$

Obviously, \mathcal{X}_κ^τ is a neighbourhood of Γ^τ in M . Assume \mathcal{X}_κ^τ carries a Riemannian metric \mathcal{H}^τ . We write \mathcal{N}^τ for the unit normal vector field on Γ^τ such that the scalar product of \mathcal{N}^τ and $d\Psi(\frac{\partial}{\partial t}, 0)$ is negative at every point of Γ^τ . In what follows, $\mathcal{H}_{\Gamma^\tau}^\tau$ is the Riemannian metric on Γ^τ induced by \mathcal{H}^τ , and $\Pi_{\Gamma^\tau}(\mathcal{H}^\tau)$ is the second fundamental form of Γ^τ in \mathcal{X}_κ^τ with respect to \mathcal{N}^τ .

For each $\tau \in [0, 1]$, consider a symmetric positive-definite G -invariant $(0, 2)$ -tensor field R^τ on Γ^τ . In order to keep our notation consistent, we assume R^0 and R^1 are the restrictions of R to Γ^0 and Γ^1 . It is evident that R^τ is fully determined by how $(\Psi(\tau, \cdot))^* R^\tau$ acts on \mathfrak{p} . There exist numbers $a_{\tau,1}, \dots, a_{\tau,n} > 0$ satisfying

$$((\Psi(\tau, \cdot))^* R^\tau)(X, Y) = a_{\tau,1}^2 Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \dots + a_{\tau,n}^2 Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \quad X, Y \in \mathfrak{p}.$$

Let us also fix, for every $\tau \in [0, 1]$, a symmetric G -invariant tensor field S^τ on Γ^τ . There are $\delta_{\tau,1}, \dots, \delta_{\tau,n} \in \mathbb{R}$ such that

$$((\Psi(\tau, \cdot))^* S^\tau)(X, Y) = \delta_{\tau,1} Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \dots + \delta_{\tau,n} Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \quad X, Y \in \mathfrak{p}.$$

Denote by $[\cdot, \cdot]$ and \mathcal{K} the Lie bracket and the Killing form of the Lie algebra \mathfrak{g} . Suppose d is the dimension of M and d_k is the dimension of \mathfrak{p}_k when $k = 1, \dots, n$. We choose a Q -orthonormal basis $(\tilde{e}_i)_{i=1}^{d-1}$ of the space \mathfrak{p} adapted to the decomposition (2.3). There exist arrays of nonnegative constants $(\beta_k)_{k=1}^n$ and $(\gamma_{k,l}^m)_{k,l,m=1}^n$ such that

$$\begin{aligned} \mathcal{K}(X, Y) &= -\beta_k Q(X, Y), \\ \sum_{\tilde{e}_i \in \mathfrak{p}_l} Q(\text{pr}_{\mathfrak{p}_m} [X, \tilde{e}_i], \text{pr}_{\mathfrak{p}_m} [X, \tilde{e}_i]) &= \gamma_{k,l}^m Q(X, X), \quad X, Y \in \mathfrak{p}_k. \end{aligned} \quad (3.1)$$

For additional information concerning these arrays, see [17] and references therein. Note that $(\beta_k)_{k=1}^n$ must contain at least one strictly positive number and $(\gamma_{k,l}^m)_{k,l,m=1}^n$ is independent of $(\tilde{e}_i)_{i=1}^{d-1}$.

Proposition 3.3, which we are about to state, underlies Theorems 3.1 and 3.2. Moreover, it demonstrates that a G -invariant metric \mathcal{H}^τ on \mathcal{X}_κ^τ solving the prescribed Ricci curvature equation $\text{Ric}(\mathcal{H}^\tau) = T$ on \mathcal{X}_κ^τ is uniquely determined by $\mathcal{H}_{\Gamma^\tau}^\tau$ and $\Pi_{\Gamma^\tau}(\mathcal{H}^\tau)$. The reader will find related material in [3, 26].

Proposition 3.3. *Suppose $\tau \in [0, 1]$. The following two statements are equivalent:*

1. *For some $\kappa \in (0, 1)$, there exists a G -invariant Riemannian metric \mathcal{H}^τ on \mathcal{X}_κ^τ such that $\text{Ric}(\mathcal{H}^\tau) = T$ on \mathcal{X}_κ^τ , $\mathcal{H}_{\Gamma^\tau}^\tau = R^\tau$, and $\Pi_{\Gamma^\tau}(\mathcal{H}^\tau) = S^\tau$.*
2. *The inequality*

$$\sum_{k=1}^n d_k \left(\frac{\beta_k}{2a_{\tau,k}^2} + \sum_{l,m=1}^n \gamma_{k,l}^m \frac{a_{\tau,k}^4 - 2a_{\tau,l}^4}{4a_{\tau,k}^2 a_{\tau,l}^2 a_{\tau,m}^2} - \sum_{l=1}^n d_l \frac{\delta_{\tau,k} \delta_{\tau,l}}{a_{\tau,k}^2 a_{\tau,l}^2} + \frac{\delta_{\tau,k}^2}{a_{\tau,k}^4} - \frac{1}{a_{\tau,k}^2} \hat{\phi}_k(\tau) \right) < 0 \quad (3.2)$$

is satisfied.

If these statements hold and $\check{\mathcal{H}}^\tau$ is a G -invariant metric on \mathcal{X}_κ^τ such that $\text{Ric}(\check{\mathcal{H}}^\tau) = T$ on \mathcal{X}_κ^τ , $\check{\mathcal{H}}_{\Gamma^\tau}^\tau = R^\tau$, and $\Pi_{\Gamma^\tau}(\check{\mathcal{H}}^\tau) = S^\tau$, then $\check{\mathcal{H}}^\tau$ must coincide with \mathcal{H}^τ .

In Section 5, we will discuss several ways to extend Proposition 3.3 as well as Theorems 3.1 and 3.2. Note that it is possible to prove these three results using the methods developed in [26]; see also [3]. We will, however, take a different approach in the present paper. In fact, one may be able to establish the theorems of [26] with the techniques employed below.

3.2 Global solvability

Our next goal is to formulate a sufficient condition for the solvability of (2.1)–(2.2) on all of M . Recall that the tensor fields T and R are given by (2.4), (2.5), and (2.6). Fix a number $\alpha > 0$ such that

$$0 < \hat{\phi}_i(t) \leq \alpha, \quad i = 1, \dots, n, \quad t \in [0, 1], \quad (3.3)$$

along with a pair of numbers $\omega_1, \omega_2 > 0$ such that

$$\omega_1 \leq a_i, b_i \leq \omega_2, \quad i = 1, \dots, n. \quad (3.4)$$

The following theorem encompasses the sufficient condition we are seeking. Remark 3.5 will briefly explain the intuition behind it.

Theorem 3.4. *There exist functions $\rho_0 : (0, \infty)^2 \rightarrow (0, \infty)$ and $\sigma_0 : (0, \infty)^3 \rightarrow (0, \infty)$, both independent of the tensor fields T and R , such that the following statement is satisfied: if the formulas*

$$\sum_{i=1}^n d_i \hat{\phi}_i(t) > \rho_0(\omega_1, \omega_2), \quad \sigma < \sigma_0(\alpha, \omega_1, \omega_2), \quad t \in [0, 1], \quad (3.5)$$

and the formulas

$$|a_i - b_i| \leq \sigma^2, \quad \left| \frac{d}{dt} \hat{\phi}_i(t) \right| \leq \sigma^2, \quad i = 1, \dots, n, \quad t \in [0, 1], \quad (3.6)$$

hold true, then the manifold M supports a G -invariant Riemannian metric \mathcal{G} solving the equation $\text{Ric}(\mathcal{G}) = T$ on M under the boundary condition $\mathcal{G}_{\partial M} = R$.

Section 5 will offer a number of variants and generalizations of this result. We will show that some of its assumptions can be changed or even eliminated.

Remark 3.5. Roughly speaking, the meaning of (3.5) is that the tensor field T has to be large in the directions tangent to the G -orbits on M and small in the direction transverse to the G -orbits. Formulas (3.6) admit analogous interpretations. The first one of them essentially says that R^0 should not be very different from R^1 . The second one forbids the part of T tangent to the G -orbits to change dramatically from one orbit to another. Note that formulas (3.6) are automatically satisfied when $(\Psi(0, \cdot))^* R^0$ coincides with $(\Psi(1, \cdot))^* R^1$ and $\hat{\phi}_1, \dots, \hat{\phi}_n$ are constant.

Remark 3.6. When proving Theorem 3.4, we will obtain explicit expressions for ρ_0 and σ_0 . These expressions (at least the one for σ_0) will be rather unsightly.

4 The proofs

It will be convenient for us to prove our results in reverse order: We will first establish Theorem 3.4. After that, we will deal with Proposition 3.3. Our last objective in Section 4 will be to derive Theorems 3.1 and 3.2 from this proposition.

4.1 Preparatory material

Let us begin by introducing some notation. Formula (2.4) can be rewritten as

$$\Psi^* T = dr \otimes dr + T_r, \quad r \in [0, \sigma].$$

The parameter r here is given by the equality $r = \sigma t$. The tensor field T_r on G/K coincides with $\hat{T}_{\frac{r}{\sigma}}$ for each $r \in [0, \sigma]$. It is easy to see that

$$T_r(X, Y) = \phi_1(r) Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \dots + \phi_n(r) Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \quad X, Y \in \mathfrak{p},$$

where $\phi_i(r) = \hat{\phi}_i(\frac{r}{\sigma})$ for $i = 1, \dots, n$.

Consider a Riemannian metric \mathcal{G} on M . Suppose h, f_1, \dots, f_n are smooth functions from $[0, \sigma]$ to $(0, \infty)$. We assume \mathcal{G} is defined by the formula

$$\Psi^* \mathcal{G} = h^2(r) dr \otimes dr + \mathcal{G}_r, \quad r \in [0, \sigma]. \quad (4.1)$$

The tensor field \mathcal{G}_r in the right-hand side is the G -invariant Riemannian metric on G/K such that

$$\mathcal{G}_r(X, Y) = f_1^2(r) Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \dots + f_n^2(r) Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \quad X, Y \in \mathfrak{p}. \quad (4.2)$$

The lemma we are about to state computes the Ricci curvature of $\Psi^* \mathcal{G}$. Note that the corresponding formula involves the arrays of constants $(\beta_k)_{k=1}^n$ and $(\gamma_{k,l}^m)_{k,l,m=1}^n$ defined by (3.1). The reader may wish to see [17] and references therein for related results. In the sequel, the prime next to a real-valued function on $[0, \sigma]$ will denote the derivative of this function.

Lemma 4.1. *The Ricci curvature of the Riemannian metric $\Psi^* \mathcal{G}$ given by (4.1) and (4.2) obeys the equality*

$$\text{Ric}(\Psi^* \mathcal{G}) = - \sum_{k=1}^n d_k \left(\frac{f_k''}{f_k} - \frac{h' f_k'}{h f_k} \right) dr \otimes dr + \mathcal{R}_r, \quad r \in [0, \sigma],$$

where \mathcal{R}_r is the G -invariant $(0, 2)$ -tensor field on G/K satisfying

$$\mathcal{R}_r(X, Y) = \sum_{i=1}^n \left(\frac{\beta_i}{2} + \sum_{k,l=1}^n \gamma_{i,k}^l \frac{f_i^4 - 2f_k^4}{4f_k^2 f_l^2} - \frac{f_i f_i'}{h} \sum_{k=1}^n d_k \frac{f_k'}{h f_k} + \frac{f_i'^2}{h^2} - \frac{f_i f_i''}{h^2} + \frac{f_i h' f_i'}{h^3} \right) Q(\text{pr}_{\mathfrak{p}_i} X, \text{pr}_{\mathfrak{p}_i} Y),$$

$X, Y \in \mathfrak{p}.$

Proof. This is a relatively simple consequence of [17, Proposition 1.14 and Remark 1.16]. \square

We need to establish one more lemma before we proceed. It is essentially a restatement of the contracted second Bianchi identity.

Lemma 4.2. *Assume the Ricci curvature of the metric $\Psi^* \mathcal{G}$ given by (4.1) and (4.2) obeys the equality*

$$\text{Ric}(\Psi^* \mathcal{G}) = \bar{\sigma}(r) dr \otimes dr + T_r, \quad r \in [0, \sigma],$$

with $\bar{\sigma}$ being a smooth function on $[0, \sigma]$. Then

$$\frac{\bar{\sigma}'}{2h^2} - \frac{\bar{\sigma}h'}{h^3} = \sum_{k=1}^n d_k \left(\frac{\phi_k'}{2f_k^2} - \frac{\bar{\sigma}f_k'}{h^2 f_k} \right).$$

Proof. Fix a Q -orthonormal basis $(\tilde{e}_i)_{i=1}^{d-1}$ of the space \mathfrak{p} adapted to the decomposition (2.3). Recall that we identify \mathfrak{p} with the tangent space of G/K at K . Given $r_0 \in [0, \sigma]$, let us construct a G -invariant $\Psi^* \mathcal{G}$ -orthonormal frame field $(e_i)_{i=1}^d$ on a neighbourhood U of $(\frac{r_0}{\sigma}, K)$ in $[0, 1] \times G/K$ so that the following requirements are met:

1. The equality $e_i = (0, \frac{1}{f_i(r)} \tilde{e}_i)$ holds at $(\frac{r}{\sigma}, K)$ for every $i = 1, \dots, d-1$ as long as $(\frac{r}{\sigma}, K) \in U$.
2. The vector field e_d coincides with $(\frac{1}{h(r)} \frac{\partial}{\partial r}, 0)$ on U .

The contracted second Bianchi identity then implies

$$\sum_{i=1}^d (\nabla_{e_i} \text{Ric}(\Psi^* \mathcal{G}))(e_i, e_d) = \frac{1}{2} e_d \left(\sum_{i=1}^d \text{Ric}(\Psi^* \mathcal{G})(e_i, e_i) \right).$$

The symbol ∇ in the left-hand side denotes the covariant derivative in the tensor bundle over $[0, 1] \times G/K$ given by the Levi-Civita connection of $\Psi^* \mathcal{G}$. We calculate and see that the equalities

$$\begin{aligned} \sum_{i=1}^d (\nabla_{e_i} \text{Ric}(\Psi^* \mathcal{G}))(e_i, e_d) &= \sum_{i=1}^d e_i(\text{Ric}(\Psi^* \mathcal{G})(e_i, e_d)) - \sum_{i=1}^d \text{Ric}(\Psi^* \mathcal{G})(\nabla_{e_i} e_i, e_d) \\ &\quad - \sum_{i=1}^d \text{Ric}(\Psi^* \mathcal{G})(e_i, \nabla_{e_i} e_d) \\ &= e_d(\text{Ric}(\Psi^* \mathcal{G})(e_d, e_d)) - \sum_{i=1}^d (\Psi^* \mathcal{G})(\nabla_{e_i} e_i, e_d) \text{Ric}(\Psi^* \mathcal{G})(e_d, e_d) \\ &\quad - \sum_{i=1}^d (\Psi^* \mathcal{G})(\nabla_{e_i} e_d, e_i) \text{Ric}(\Psi^* \mathcal{G})(e_i, e_i) \\ &= \frac{\bar{\sigma}'}{h^3} - \frac{2\bar{\sigma}h'}{h^4} + \sum_{k=1}^n d_k \frac{\bar{\sigma}f_k'}{h^3 f_k} - \sum_{k=1}^n d_k \frac{f_k'}{h f_k^3} \phi_k, \end{aligned}$$

as well as the equality

$$\frac{1}{2} e_d \left(\sum_{i=1}^d \text{Ric}(\Psi^* \mathcal{G})(e_i, e_i) \right) = \sum_{k=1}^n d_k \left(\frac{\phi_k'}{2h f_k^2} - \frac{f_k'}{h f_k^3} \phi_k \right) + \frac{\bar{\sigma}'}{2h^3} - \frac{\bar{\sigma}h'}{h^4},$$

hold at $(\frac{r_0}{\sigma}, K)$. The assertion of the lemma follows immediately. \square

Let us make a few more computations. After doing so, we will lay out our strategy for proving Theorem 3.4. If the Ricci curvature of \mathcal{G} coincides with T , then Lemma 4.1 yields the equalities

$$-\sum_{k=1}^n d_k \left(\frac{f_k''}{f_k} - \frac{h' f_k'}{h f_k} \right) = 1,$$

$$\frac{\beta_i}{2} + \sum_{k,l=1}^n \gamma_{i,k}^l \frac{f_i^4 - 2f_k^4}{4f_k^2 f_l^2} - \frac{f_i f_i'}{h} \sum_{k=1}^n d_k \frac{f_k'}{h f_k} + \frac{f_i'^2}{h^2} - \frac{f_i f_i''}{h^2} + \frac{f_i h' f_i'}{h^3} = \phi_i, \quad i = 1, \dots, n. \quad (4.3)$$

Consequently, we have

$$H_1(f(r), f'(r)) = h^2(r) H_2(f(r), \phi(r)), \quad r \in [0, \sigma], \quad (4.4)$$

with the mappings $H_1 : (0, \infty)^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $H_2 : (0, \infty)^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ defined by the formulas

$$H_1(x, y) = 1 - \sum_{k=1}^n d_k \left(\sum_{l=1}^n d_l \frac{y_k y_l}{x_k x_l} - \frac{y_k^2}{x_k^2} \right), \quad H_2(x, z) = \sum_{k=1}^n d_k \left(\frac{z_k}{x_k^2} - \frac{\beta_k}{2x_k^2} - \sum_{l,m=1}^n \gamma_{k,l}^m \frac{x_k^4 - 2x_l^4}{4x_k^2 x_l^2 x_m^2} \right),$$

$$x = (x_1, \dots, x_n) \in (0, \infty)^n, \quad y = (y_1, \dots, y_n) \in \mathbb{R}^n, \quad z = (z_1, \dots, z_n) \in \mathbb{R}^n.$$

The letters f and ϕ in (4.4) denote the functions (f_1, \dots, f_n) and (ϕ_1, \dots, ϕ_n) from $[0, \sigma]$ to $(0, \infty)^n$ and \mathbb{R}^n , respectively. The prime means component-wise differentiation. Along with (4.4), equalities (4.3) imply

$$f''(r) = F(h(r), h'(r), f(r), f'(r), \phi(r)), \quad r \in [0, \sigma], \quad (4.5)$$

with $F : (0, \infty) \times \mathbb{R} \times (0, \infty)^n \times \mathbb{R}^{n+n} \rightarrow \mathbb{R}^n$ given by the formulas

$$F(p, q, x, y, z) = (F_1(p, q, x, y, z), \dots, F_n(p, q, x, y, z)),$$

$$F_i(p, q, x, y, z) = \frac{\beta_i p^2}{2x_i} + p^2 \sum_{k,l=1}^n \gamma_{i,k}^l \frac{x_i^4 - 2x_k^4}{4x_i x_k^2 x_l^2} - \sum_{k=1}^n d_k \frac{y_i y_k}{x_k} + \frac{y_i^2}{x_i} + \frac{q y_i}{p} - \frac{p^2}{x_i} z_i, \quad i = 1, \dots, n,$$

$$p \in (0, \infty), \quad q \in \mathbb{R}, \quad x = (x_1, \dots, x_n) \in (0, \infty)^n, \quad y = (y_1, \dots, y_n) \in \mathbb{R}^n, \quad z = (z_1, \dots, z_n) \in \mathbb{R}^n.$$

According to Lemma 4.2, if $\text{Ric}(\mathcal{G})$ coincides with T , then

$$h'(r) = K(h(r), f(r), f'(r), \phi'(r)), \quad r \in [0, \sigma]. \quad (4.6)$$

Here, $K : (0, \infty)^{1+n} \times \mathbb{R}^{n+n} \rightarrow \mathbb{R}$ is given by

$$K(p, x, y, w) = \sum_{i=1}^n d_i \left(\frac{p y_i}{x_i} - \frac{p^3 w_i}{2x_i^2} \right),$$

$$p \in (0, \infty), \quad x = (x_1, \dots, x_n) \in (0, \infty)^n, \quad y = (y_1, \dots, y_n) \in \mathbb{R}^n, \quad w = (w_1, \dots, w_n) \in \mathbb{R}^n.$$

Let a and b denote the vectors (a_1, \dots, a_n) and (b_1, \dots, b_n) with the numbers a_1, \dots, a_n and b_1, \dots, b_n coming from (2.6). If the metric $\mathcal{G}_{\partial M}$ induced by \mathcal{G} on ∂M equals R , then

$$f(0) = a, \quad f(\sigma) = b. \quad (4.7)$$

We also point out that, whenever (4.4) holds, we must have

$$H_1(f(0), f'(0)) = h^2(0) H_2(f(0), \phi(0)). \quad (4.8)$$

4.2 Proof of Theorem 3.4 (less the key lemma)

Our strategy for proving Theorem 3.4 is to produce smooth functions f and h satisfying equations (4.5)–(4.6) together with the boundary conditions (4.7)–(4.8). Using these functions, we will then define the metric \mathcal{G}

through formulas (4.1)–(4.2). In the end, we will demonstrate that the Ricci curvature of this metric coincides with T .

Intuitively, the plan is to find \mathcal{G} satisfying two requirements. The first one is that $\text{Ric}(\mathcal{G})$ equals T in the directions tangent to the G -orbits. The other is that \mathcal{G} and T obey the contracted second Bianchi identity. When both of these requirements are met, it must be the case that $\text{Ric}(\mathcal{G}) = T$.

Given $p, q \in (0, \infty)$, define $\rho_0(p, q)$ by the formula

$$\rho_0(p, q) = 2 \sum_{k=1}^n d_k \left(\frac{\beta_k q^2}{2p^2} + \sum_{l,m=1}^n \gamma_{k,l}^m \frac{q^6}{4p^6} \right).$$

Lemma 4.3. *Assume the first inequality in (3.5) and inequalities (3.6) are satisfied. There exists a function $\sigma_0 : (0, \infty)^3 \rightarrow (0, \infty)$ such that the following statement holds: if σ is less than $\sigma_0(\alpha, \omega_1, \omega_2)$, then we can find smooth $f : [0, \sigma] \rightarrow (0, \infty)^n$ and $h : [0, \sigma] \rightarrow (0, \infty)$ solving equations (4.5)–(4.6) under the boundary conditions (4.7)–(4.8).*

The proof of Lemma 4.3 is rather lengthy and technically involved. We will present it in Section 4.3. Meanwhile, fix a function σ_0 satisfying the assertion of this lemma. Suppose σ is less than $\sigma_0(\alpha, \omega_1, \omega_2)$. Let $f : [0, \sigma] \rightarrow (0, \infty)^n$ and $h : [0, \sigma] \rightarrow (0, \infty)$ be smooth functions obeying (4.5)–(4.6) and (4.7)–(4.8). We define the metric \mathcal{G} on M through (4.1)–(4.2). It is easy to see that the Ricci curvature of $\Psi^*\mathcal{G}$ must equal

$$\bar{\sigma}(r) dr \otimes dr + T_r, \quad r \in [0, \sigma],$$

for some $\bar{\sigma} : [0, \sigma] \rightarrow \mathbb{R}$. The induced metric $\mathcal{G}_{\partial M}$ coincides with R . Let us denote by I the function on $[0, \sigma]$ such that $I(r) = 1$ for all $r \in [0, \sigma]$. The proof of Theorem 3.4 will be complete if we demonstrate that $\bar{\sigma} = I$.

Consider the equation

$$v' = \frac{2vh'}{h} + \sum_{i=1}^n d_i \left(\frac{h^2 \phi'_i}{f_i^2} - \frac{2vf'_i}{f_i} \right) \quad (4.9)$$

for the unknown $v : [0, \sigma] \rightarrow \mathbb{R}$. Lemma 4.2 implies that $\bar{\sigma}$ solves (4.9). Formula (4.6) tells us that I satisfies (4.9) as well. Furthermore, invoking Lemma 4.1 and the boundary conditions (4.7)–(4.8), we find

$$\begin{aligned} \bar{\sigma}(0) &= - \sum_{k=1}^n d_k \left(\frac{f''_k(0)}{a_k} - \frac{h'(0)f'_k(0)}{h(0)a_k} \right) \\ &= - \sum_{k=1}^n d_k \left(\frac{\beta_k h^2(0)}{2a_k^2} + h^2(0) \sum_{l,m=1}^n \gamma_{k,l}^m \frac{a_k^4 - 2a_l^4}{4a_k^2 a_l^2 a_m^2} - \sum_{l=1}^n d_l \frac{f'_k(0)f'_l(0)}{a_k a_l} + \frac{f_k'^2(0)}{a_k^2} - \frac{h^2(0)}{a_k^2} \phi_k(0) \right) \\ &= h^2(0)H_2(a, \phi(0)) + (1 - H_1(a, f'(0))) = 1 = I(0). \end{aligned}$$

It becomes clear that $\bar{\sigma} = I$.

4.3 Proof of Lemma 4.3

Section 4.2 produced a metric \mathcal{G} on M such that $\text{Ric}(\mathcal{G}) = T$ and $\mathcal{G}_{\partial M} = R$. However, we left a substantial gap in our reasoning. Namely, we did not present a proof of Lemma 4.3. The purpose of Section 4.3 is to fill in this gap. We will prove Lemma 4.3 using the Schauder fixed point theorem. The reader may see, e.g., [19, Chapter XII] for the relevant background material. From now on and until the end of Section 4.3, we assume the first inequality in (3.5) and inequalities (3.6) are satisfied.

It is appropriate to begin with two more pieces of notation. Let \bar{f} be the function from $[0, \sigma]$ to \mathbb{R}^n defined by

$$\bar{f}(r) = a \frac{\sigma - r}{\sigma} + b \frac{r}{\sigma}, \quad r \in [0, \sigma].$$

Given $x \in (0, \infty)^n$, $y \in \mathbb{R}^n$, and $z \in \mathbb{R}^n$ such that $H_2(x, z) \neq 0$ and $H_1(x, y)H_2^{-1}(x, z) \geq 0$, we write $H(x, y, z)$ for the quantity

$$\sqrt{H_1(x, y)H_2^{-1}(x, z)}.$$

Also, recall that the letter d stands for the dimension of M . It is evident that $\sum_{i=1}^n d_i = d - 1$. We have the following auxiliary result.

Lemma 4.4. *Let $\rho_1, \sigma_1 > 0$ be given by the formulas*

$$\begin{aligned} \rho_1 &= \max \left\{ 4 \left(\sum_{k=1}^n d_k \left(\frac{\alpha}{\omega_1^2} + \sum_{l,m=1}^n \gamma_{k,l}^m \frac{\omega_2^4}{2\omega_1^6} \right) \right)^{\frac{1}{2}}, \frac{9}{4} \left(\frac{\rho_0(\omega_1, \omega_2)}{2\omega_2^2} \right)^{-\frac{1}{2}} \right\}, \\ \sigma_1 &= \min \left\{ 1, \frac{\omega_1}{4d}, \frac{2\omega_1^2}{(2\rho_1^2\omega_1 + \rho_1^4)(d-1)} \right\}. \end{aligned}$$

If $\sigma \leq \sigma_1$, then

$$\begin{aligned} H_2(\bar{f}(0), \phi(0)) &\neq 0, \\ H_1(\bar{f}(0), \bar{f}'(0))H_2^{-1}(\bar{f}(0), \phi(0)) &> 0. \end{aligned} \tag{4.10}$$

Moreover, in this case, the problem

$$\begin{aligned} \bar{h}'(r) &= K(\bar{h}(r), \bar{f}(r), \bar{f}'(r), \phi'(r)), \quad r \in [0, \sigma], \\ \bar{h}(0) &= H(\bar{f}(0), \bar{f}'(0), \phi(0)), \end{aligned} \tag{4.11}$$

has a unique smooth solution $\bar{h} : [0, \sigma] \rightarrow \left(\frac{1}{\rho_1}, \rho_1\right)$.

Proof. Assume $\sigma < \sigma_1$. Formulas (4.10) follow from the first inequalities in (3.5) and (3.6). Let us denote $H(\bar{f}(0), \bar{f}'(0), \phi(0))$ by H_0 . Suppose K_0 is a positive number such that

$$\sup_{p \in [\frac{H_0}{2}, \frac{3H_0}{2}]} \sup_{r \in [0, \sigma]} |K(p, \bar{f}(r), \bar{f}'(r), \phi'(r))| \leq K_0.$$

Employing the standard theory of ordinary differential equations (specifically, the Picard-Lindelöf theorem; see, e.g., [19, Chapter II]), it is easy to show that problem (4.11) has a unique smooth solution on the interval $\left[0, \min \left\{ \sigma, \frac{H_0}{2K_0} \right\} \right]$. The values of this solution must lie in $\left[\frac{H_0}{2}, \frac{3H_0}{2}\right]$.

Our assumptions imply

$$\frac{1}{\rho_1} \leq \frac{H_0}{2} < \frac{3H_0}{2} \leq \rho_1.$$

In view of (3.6), the estimate

$$\sup_{p \in [\frac{H_0}{2}, \frac{3H_0}{2}]} \sup_{r \in [0, \sigma]} |K(p, \bar{f}(r), \bar{f}'(r), \phi'(r))| \leq \frac{(2\rho_1\omega_1\sigma + \rho_1^3\sigma)(d-1)}{2\omega_1^2}$$

holds true. Keeping these facts in mind, we conclude that problem (4.11) has a unique smooth solution

$$\bar{h} : \left[0, \min \left\{ \sigma, \frac{2\omega_1^2}{(2\rho_1^2\omega_1\sigma + \rho_1^4\sigma)(d-1)} \right\} \right] \rightarrow \left(\frac{1}{\rho_1}, \rho_1\right).$$

At the same time, whenever $\sigma \leq \sigma_1$, the equality

$$\sigma = \min \left\{ \sigma, \frac{2\omega_1^2}{(2\rho_1^2\omega_1\sigma + \rho_1^4\sigma)(d-1)} \right\}$$

is satisfied. This means \bar{h} is actually defined on $[0, \sigma]$. □

Our goal is to produce, for sufficiently small σ , smooth functions f and h on $[0, \sigma]$ obeying (4.5)–(4.6) and (4.7)–(4.8). From this moment on and until the end of Section 4.3, let us assume that $\sigma \leq \sigma_1$. It then makes sense to talk about \bar{h} . Our plan is to prove, for small σ , the existence of smooth $u : [0, \sigma] \rightarrow \mathbb{R}^n$ and $v : [0, \sigma] \rightarrow \mathbb{R}$ solving the equations

$$\begin{aligned} u''(r) &= F(\bar{h}(r) + v(r), \bar{h}'(r) + v'(r), \bar{f}(r) + u(r), \bar{f}'(r) + u'(r), \phi(r)), \\ v'(r) &= -\bar{h}'(r) + K(\bar{h}(r) + v(r), \bar{f}(r) + u(r), \bar{f}'(r) + u'(r), \phi'(r)), \quad r \in [0, \sigma], \end{aligned} \quad (4.12)$$

under the boundary conditions

$$\begin{aligned} u(0) &= u(\sigma) = 0, \\ v(0) &= -\bar{h}(0) + H(\bar{f}(0) + u(0), \bar{f}'(0) + u'(0), \phi(0)). \end{aligned} \quad (4.13)$$

We will then set $f = \bar{f} + u$ and $h = \bar{h} + v$. It is obvious that these functions will obey (4.5)–(4.6) and (4.7)–(4.8).

Our proof of the existence of u and v will rely on the Schauder fixed point theorem. Let us introduce the space \mathcal{B} of all the pairs (v_1, v_2) such that $v_1 : [0, \sigma] \rightarrow \mathbb{R}^n$ is C^1 -differentiable and $v_2 : [0, \sigma] \rightarrow \mathbb{R}$ is continuous. We endow \mathcal{B} with the norm

$$|(v_1, v_2)|_{\mathcal{B}} = \sup_{r \in [0, \sigma]} |v_1(r)|_{\mathbb{R}^n} + \sigma \sup_{r \in [0, \sigma]} |v_1'(r)|_{\mathbb{R}^n} + \sup_{r \in [0, \sigma]} |v_2(r)|,$$

where $|\cdot|_{\mathbb{R}^n}$ is the Euclidean norm in \mathbb{R}^n . Denote by B the closed ball in \mathcal{B} of radius $L > 0$ centered at 0. We will now define a map $\mathcal{C} : B \rightarrow \mathcal{B}$ and show that \mathcal{C} has a fixed point (u, v) under appropriate conditions. The functions u and v will satisfy (4.12) and (4.13).

Assume the radius L is less than or equal to $\frac{\sigma}{2} \min\{\omega_1, \frac{1}{\rho_1}\}$. Given $(\mu, \nu) \in B$, let $\xi_{\mu, \nu}$ be the unique solution of the problem

$$\begin{aligned} \xi_{\mu, \nu}''(r) &= \tilde{F}(\bar{h}(r) + \nu(r), \bar{f}(r) + \mu(r), \bar{f}'(r) + \mu'(r), \phi(r), \phi'(r)), \quad r \in [0, \sigma], \\ \xi_{\mu, \nu}(0) &= \xi_{\mu, \nu}(\sigma) = 0, \end{aligned} \quad (4.14)$$

with

$$\tilde{F}(p, x, y, z, w) = F(p, K(p, x, y, w), x, y, z), \quad p \in (0, \infty), \quad x \in (0, \infty)^n, \quad y, z, w \in \mathbb{R}^n;$$

see, e.g., [19, Section XII.4]. We will set $\mathcal{C}(\mu, \nu) = (\xi_{\mu, \nu}, \zeta_{\mu, \nu})$ for a properly chosen $\zeta_{\mu, \nu} : [0, \sigma] \rightarrow \mathbb{R}$. Before we can describe $\zeta_{\mu, \nu}$, however, we need to state the following auxiliary result.

Lemma 4.5. *Let Θ be given by the formulas*

$$\begin{aligned} \Theta &= |(\Theta_1, \dots, \Theta_n)|_{\mathbb{R}^n}, \\ \Theta_i &= \frac{4\beta_i \rho_1^2}{\omega_1} + 1, 536 \rho_1^2 \sum_{k, l=1}^n \gamma_{i, k}^l \frac{\omega_2^4}{\omega_1^5} + 2\omega_1 + (2\omega_1 + 2\omega_1^2 + 8\rho_1^2)(d-1) + \frac{8\alpha \rho_1^2}{\omega_1}, \quad i = 1, \dots, n. \end{aligned}$$

If (μ, ν) lie in B , then the estimate

$$\sup_{r \in [0, \sigma]} |\tilde{F}(\bar{h}(r) + \nu(r), \bar{f}(r) + \mu(r), \bar{f}'(r) + \mu'(r), \phi(r), \phi'(r))|_{\mathbb{R}^n} \leq \Theta \quad (4.15)$$

holds true. Moreover, in this case, we have

$$|\xi_{\mu, \nu}(r)|_{\mathbb{R}^n} \leq \frac{\sigma^2}{8} \Theta, \quad |\xi'_{\mu, \nu}(r)|_{\mathbb{R}^n} \leq \frac{\sigma}{2} \Theta, \quad r \in [0, \sigma]. \quad (4.16)$$

Proof. Estimate (4.15) is a straightforward consequence of the definition of \tilde{F} . Formulas (4.16) follow from the arguments in [19, Section XII.4]. \square

Let us denote by ϵ_0 the number $\frac{\omega_1}{2d}$. It is not difficult to check that the expression $H(\bar{f}(0), y, \phi(0))$ is well-defined and positive whenever $|y|_{\mathbb{R}^n} \leq \epsilon_0$. From now on and until the end of Section 4.3, we assume

$$\sigma \leq \min \left\{ \sigma_1, \sqrt{\frac{\omega_1}{\Theta}}, \frac{\epsilon_0}{\Theta} \right\}. \quad (4.17)$$

Given $(\mu, \nu) \in B$, let us introduce $\zeta_{\mu, \nu} : [0, \sigma] \rightarrow \mathbb{R}$ through the formula

$$\begin{aligned} \zeta_{\mu, \nu}(r) = & -\bar{h}(0) + H(\bar{f}(0) + \xi_{\mu, \nu}(0), \bar{f}'(0) + \xi'_{\mu, \nu}(0), \phi(0)) \\ & + \int_0^r \left(-\bar{h}'(s) + K(\bar{h}(s) + \nu(s), \bar{f}(s) + \xi_{\mu, \nu}(s), \bar{f}'(s) + \xi'_{\mu, \nu}(s), \phi'(s)) \right) ds, \quad r \in [0, \sigma]. \end{aligned} \quad (4.18)$$

Lemma 4.5 and inequality (4.17) imply the estimates

$$\sup_{r \in [0, \sigma]} |\xi_{\mu, \nu}(r)|_{\mathbb{R}^n} \leq \frac{\omega_1}{2}, \quad \sup_{r \in [0, \sigma]} |\xi'_{\mu, \nu}(r)|_{\mathbb{R}^n} \leq \frac{\epsilon_0}{2},$$

which ensure that the right-hand side of (4.18) is well-defined. We now set $\mathcal{C}(\mu, \nu) = (\xi_{\mu, \nu}, \zeta_{\mu, \nu})$. Our intention is to demonstrate that, when σ is sufficiently small and the radius L is appropriately chosen, the map \mathcal{C} has a fixed point. The first step is to show that, for such σ and L , the image $\mathcal{C}B$ is a subset of B .

A few more pieces of notation are required. Suppose $\theta_1 > 0$ is a constant obeying the inequalities

$$\begin{aligned} |H(x, y, z) - H(x, \hat{y}, z)| & \leq \theta_1 |y - \hat{y}|_{\mathbb{R}^n}, \\ |H(x, y, z) - H(x, \hat{y}, z)| & \leq \theta_1 \sum_{k, l=1}^n |y_k y_l - \hat{y}_k \hat{y}_l|, \\ x & \in [\omega_1, \omega_2]^n, \quad y = (y_1, \dots, y_n) \in [-\epsilon_0, \epsilon_0]^n, \quad \hat{y} = (\hat{y}_1, \dots, \hat{y}_n) \in [-\epsilon_0, \epsilon_0]^n, \\ z & \in \left\{ (z_1, \dots, z_n) \in [0, \alpha]^n \left| \sum_{k=1}^n d_k z_k \geq \rho_0(\omega_1, \omega_2) \right. \right\}, \end{aligned} \quad (4.19)$$

and $\theta_2 > 0$ is a constant satisfying

$$\begin{aligned} |K(p, x, y, w) - K(\hat{p}, \hat{x}, \hat{y}, w)| & \leq \theta_2 (|p - \hat{p}| + |x - \hat{x}|_{\mathbb{R}^n} + |y - \hat{y}|_{\mathbb{R}^n}), \\ p, \hat{p} & \in \left[\frac{1}{2\rho_1}, 2\rho_1 \right], \quad x, \hat{x} \in \left[\frac{\omega_1}{2}, 2\omega_2 \right]^n, \quad y, \hat{y} \in [-\epsilon_0, \epsilon_0]^n, \quad w \in [-1, 1]^n. \end{aligned} \quad (4.20)$$

It is obvious that such θ_1 and θ_2 exist. We define

$$\begin{aligned} \Sigma & = \Theta + \theta_1 n^2 (\Theta + \Theta^2) + \theta_2 (\omega_1 + \Theta), \\ \sigma_0(\alpha, \omega_1, \omega_2) & = \min \left\{ \sigma_1, \sqrt{\frac{\omega_1}{\Theta}}, \frac{\epsilon_0}{\Theta}, \frac{\omega_1}{2\Sigma}, \frac{1}{2\rho_1 \Sigma} \right\}. \end{aligned}$$

Let us also set $L = \sigma^2 \Sigma$. From now on and until the end of this section, we will assume the second inequality in (3.5) holds. This implies, in particular, that L cannot exceed $\frac{\sigma}{2} \min \left\{ \omega_1, \frac{1}{\rho_1} \right\}$.

Lemma 4.6. *The image $\mathcal{C}B$ is contained in B .*

Proof. Take a pair (μ, ν) from B . Our goal is to show that $\mathcal{C}(\mu, \nu)$ lies in B . Clearly, it would suffice to prove that $|(\xi_{\mu, \nu}, \zeta_{\mu, \nu})|_{\mathcal{B}}$ is less than or equal to $\sigma^2 \Sigma$. Lemma 4.5 yields the estimate

$$|(\xi_{\mu, \nu}, \zeta_{\mu, \nu})|_{\mathcal{B}} \leq \sigma^2 \Theta + \sup_{r \in [0, \sigma]} |\zeta_{\mu, \nu}(r)|.$$

Remembering the first formula in (3.6), we also find

$$\begin{aligned}
|\zeta_{\mu,\nu}(r)| &\leq \left| -\bar{h}(0) + H(\bar{f}(0) + \xi_{\mu,\nu}(0), \bar{f}'(0) + \xi'_{\mu,\nu}(0), \phi(0)) \right| \\
&\quad + \sigma \sup_{s \in [0,r]} \left| -\bar{h}'(s) + K(\bar{h}(s) + \nu(s), \bar{f}(s) + \xi_{\mu,\nu}(s), \bar{f}'(s) + \xi'_{\mu,\nu}(s), \phi'(s)) \right| \\
&= \left| H(\bar{f}(0), \bar{f}'(0) + \xi'_{\mu,\nu}(0), \phi(0)) - H(\bar{f}(0), \bar{f}'(0), \phi(0)) \right| \\
&\quad + \sigma \sup_{s \in [0,r]} \left| K(\bar{h}(s) + \nu(s), \bar{f}(s) + \xi_{\mu,\nu}(s), \bar{f}'(s) + \xi'_{\mu,\nu}(s), \phi'(s)) - K(\bar{h}(s), \bar{f}(s), \bar{f}'(s), \phi'(s)) \right| \\
&\leq \theta_1 \sum_{k,l=1}^n (\sigma |(\xi_{\mu,\nu})'_k(0)| + \sigma |(\xi_{\mu,\nu})'_l(0)| + |(\xi_{\mu,\nu})'_k(0)(\xi_{\mu,\nu})'_l(0)|) \\
&\quad + \sigma \theta_2 \sup_{s \in [0,r]} (|\nu(s)| + |\xi_{\mu,\nu}(s)|_{\mathbb{R}^n} + |\xi'_{\mu,\nu}(s)|_{\mathbb{R}^n}) \\
&\leq \sigma^2 \theta_1 n^2 (\Theta + \Theta^2) + \sigma^2 \theta_2 (\omega_1 + \Theta), \quad r \in [0, \sigma],
\end{aligned}$$

where $(\xi_{\mu,\nu})_k$ and $(\xi_{\mu,\nu})_l$ are the k th and the l th components of $\xi_{\mu,\nu}$. Consequently, it must be the case that

$$|(\xi_{\mu,\nu}, \zeta_{\mu,\nu})|_{\mathcal{B}} \leq \sigma^2 (\Theta + \theta_1 n^2 (\Theta + \Theta^2) + \theta_2 (\omega_1 + \Theta)) = \sigma^2 \Sigma.$$

□

Our objective is to prove the existence of $(u, v) \in B$ satisfying the equality $\mathcal{C}(u, v) = (u, v)$. The plan is to apply the Schauder fixed point theorem to \mathcal{C} . Before we can do so, however, we have to verify that \mathcal{C} is continuous. Once that is done, we will also need to check that $\mathcal{C}B$ is precompact.

Lemma 4.7. *The map $\mathcal{C} : B \rightarrow \mathcal{B}$ is continuous.*

Proof. Fix a constant $\theta_3 > 0$ such that

$$\begin{aligned}
|\tilde{F}(p, x, y, z, w) - \tilde{F}(\hat{p}, \hat{x}, \hat{y}, z, w)|_{\mathbb{R}^n} &\leq \theta_3 (|p - \hat{p}| + |x - \hat{x}|_{\mathbb{R}^n} + |y - \hat{y}|_{\mathbb{R}^n}), \\
p, \hat{p} &\in \left[\frac{1}{2\rho_1}, 2\rho_1 \right], \quad x, \hat{x} \in \left[\frac{\omega_1}{2}, 2\omega_2 \right]^n, \quad y, \hat{y} \in \left[-\frac{\omega_1}{2}, \frac{\omega_1}{2} \right]^n, \quad z \in [0, \alpha]^n, \quad w \in [-1, 1]^n. \quad (4.21)
\end{aligned}$$

Suppose the pairs (μ_1, ν_1) and (μ_2, ν_2) lie in B . The first formula in (4.14), the arguments in [19, Section XII.4], and inequality (4.21) imply

$$\begin{aligned}
\sup_{r \in [0, \sigma]} |\xi_{\mu_1, \nu_1}(r) - \xi_{\mu_2, \nu_2}(r)|_{\mathbb{R}^n} &\leq \frac{\sigma \theta_3}{8} |(\mu_1, \nu_1) - (\mu_2, \nu_2)|_{\mathcal{B}}, \\
\sup_{r \in [0, \sigma]} |\xi'_{\mu_1, \nu_1}(r) - \xi'_{\mu_2, \nu_2}(r)|_{\mathbb{R}^n} &\leq \frac{\theta_3}{2} |(\mu_1, \nu_1) - (\mu_2, \nu_2)|_{\mathcal{B}}.
\end{aligned}$$

Using (4.18), (4.19), and (4.20), we also find

$$\begin{aligned}
\sup_{r \in [0, \sigma]} |\zeta_{\mu_1, \nu_1}(r) - \zeta_{\mu_2, \nu_2}(r)| &\leq \theta_1 |\xi'_{\mu_1, \nu_1}(0) - \xi'_{\mu_2, \nu_2}(0)|_{\mathbb{R}^n} + \theta_2 \int_0^\sigma |\nu_1(s) - \nu_2(s)| ds \\
&\quad + \theta_2 \int_0^\sigma (|\xi_{\mu_1, \nu_1}(s) - \xi_{\mu_2, \nu_2}(s)|_{\mathbb{R}^n} + |\xi'_{\mu_1, \nu_1}(s) - \xi'_{\mu_2, \nu_2}(s)|_{\mathbb{R}^n}) ds \\
&\leq \left(\frac{\theta_1 \theta_3}{2} + \sigma \theta_2 + \sigma \theta_2 \theta_3 \right) |(\mu_1, \nu_1) - (\mu_2, \nu_2)|_{\mathcal{B}}.
\end{aligned}$$

Consequently, it must be the case that

$$|\mathcal{C}(\mu_1, \nu_1) - \mathcal{C}(\mu_2, \nu_2)|_{\mathcal{B}} \leq \left(\frac{\theta_1 \theta_3}{2} + \sigma \theta_2 + \sigma \theta_3 + \sigma \theta_2 \theta_3 \right) |(\mu_1, \nu_1) - (\mu_2, \nu_2)|_{\mathcal{B}},$$

which tells us \mathcal{C} is continuous.

□

It remains to check one last thing before the Schauder fixed point theorem can be applied. Namely, we need to demonstrate that \mathcal{CB} is precompact. In order to do so, we will utilize the Arzelà-Ascoli theorem.

Lemma 4.8. *The closure of the set \mathcal{CB} in \mathcal{B} is a compact subset of \mathcal{B} .*

Proof. Suppose $((\mu_j, \nu_j))_{j=1}^\infty$ are pairs from B . It suffices to prove that the sequence $((\xi_{\mu_j, \nu_j}, \zeta_{\mu_j, \nu_j}))_{j=1}^\infty$ has a convergent subsequence. The mean value theorem and Lemma 4.5 yield the estimates

$$\begin{aligned} |\xi_{\mu_j, \nu_j}(r_1) - \xi_{\mu_j, \nu_j}(r_2)|_{\mathbb{R}^n} &\leq \sup_{r \in [0, \sigma]} |\xi'_{\mu_j, \nu_j}(r)|_{\mathbb{R}^n} |r_1 - r_2| \leq \frac{\sigma}{2} \Theta |r_1 - r_2|, \\ |\xi'_{\mu_j, \nu_j}(r_1) - \xi'_{\mu_j, \nu_j}(r_2)|_{\mathbb{R}^n} &\leq \sup_{r \in [0, \sigma]} |\xi''_{\mu_j, \nu_j}(r)|_{\mathbb{R}^n} |r_1 - r_2| \leq \Theta |r_1 - r_2|, \quad j = 1, 2, \dots, \end{aligned}$$

for $r_1, r_2 \in [0, \sigma]$. Recalling formulas (4.11) and (4.20), we also obtain

$$\begin{aligned} |\zeta_{\mu_j, \nu_j}(r_1) - \zeta_{\mu_j, \nu_j}(r_2)| &\leq \sup_{r \in [0, \sigma]} |\zeta'_{\mu_j, \nu_j}(r)| |r_1 - r_2| \\ &\leq \sup_{r \in [0, \sigma]} |-\bar{h}'(r) + K(\bar{h}(r) + \nu_j(r), \bar{f}(r) + \xi_{\mu_j, \nu_j}(r), \bar{f}'(r) + \xi'_{\mu_j, \nu_j}(r), \phi'(r))| |r_1 - r_2| \\ &\leq \theta_2 \sup_{r \in [0, \sigma]} (|\nu_j(r)| + |\xi_{\mu_j, \nu_j}(r)|_{\mathbb{R}^n} + |\xi'_{\mu_j, \nu_j}(r)|_{\mathbb{R}^n}) |r_1 - r_2| \\ &\leq \theta_2 (\sigma^2 \Sigma + \sigma \Theta) |r_1 - r_2|, \quad j = 1, 2, \dots, \quad r_1, r_2 \in [0, \sigma]. \end{aligned}$$

It follows that the sequences $(\xi_{\mu_j, \nu_j})_{j=1}^\infty$, $(\xi'_{\mu_j, \nu_j})_{j=1}^\infty$, and $(\zeta_{\mu_j, \nu_j})_{j=1}^\infty$ are equicontinuous. Furthermore, because \mathcal{CB} is a subset of B , they are uniformly bounded. These facts, along with the Arzelà-Ascoli theorem, imply that $((\xi_{\mu_j, \nu_j}, \zeta_{\mu_j, \nu_j}))_{j=1}^\infty$ must have a convergent subsequence. \square

The proof of Lemma 4.3 is almost finished. As we have shown above, the map $\mathcal{C} : B \rightarrow \mathcal{B}$ is continuous, and its image is a precompact subset of B . Keeping this in mind and applying the Schauder fixed point theorem (see [19, Chapter XII, Corollary 0.1]), we conclude that there exists a pair $(u, v) \in B$ satisfying the equality $\mathcal{C}(u, v) = (u, v)$. It is easy to understand that u and v obey (4.12) and (4.13). A simple bootstrapping argument demonstrates that u and v are smooth. We define $f = \bar{f} + u$ and $h = \bar{h} + v$. Clearly, these functions take values in $(0, \infty)^n$ and $(0, \infty)$, respectively, and solve (4.5)–(4.6) under the conditions (4.7)–(4.8). Thus, Lemma 4.3 is established.

4.4 Proof of Proposition 3.3

Suppose there exist $\kappa \in (0, 1)$ and a G -invariant Riemannian metric \mathcal{H}^τ on \mathcal{X}_κ^τ such that $\text{Ric}(\mathcal{H}^\tau) = T$ on \mathcal{X}_κ^τ , $\mathcal{H}_{\Gamma^\tau}^\tau = R^\tau$, and $\Pi_{\Gamma^\tau}(\mathcal{H}^\tau) = S^\tau$. Employing Lemma 4.1 and the fact that T is positive-definite, one can show that \mathcal{H}^τ satisfies the formula

$$\Psi^* \mathcal{H}^\tau = h_\tau^2(r) dr \otimes dr + \mathcal{H}_r^\tau, \quad r \in J_\kappa^\tau = (\sigma(\tau - \kappa), \sigma(\tau + \kappa)) \cap [0, \sigma]. \quad (4.22)$$

Here, h_τ is a smooth function acting from J_κ^τ to $(0, \infty)$. The tensor field \mathcal{H}_r^τ is a G -invariant Riemannian metric on G/K . It is clear that

$$\mathcal{H}_r^\tau(X, Y) = f_{\tau, 1}^2(r) Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \dots + f_{\tau, n}^2(r) Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \quad X, Y \in \mathfrak{p}, \quad (4.23)$$

for some smooth functions $f_{\tau, 1}, \dots, f_{\tau, n}$ from J_κ^τ to $(0, \infty)$. The equality $\text{Ric}(\mathcal{H}^\tau) = T$ and Lemma 4.1 imply

$$H_1(f_\tau(r), f'_\tau(r)) = h_\tau^2(r) H_2(f_\tau(r), \phi(r)), \quad r \in J_\kappa^\tau. \quad (4.24)$$

The notation f_τ here stands for $(f_{\tau, 1}, \dots, f_{\tau, n})$. Because $\mathcal{H}_{\Gamma^\tau}^\tau = R^\tau$ and $\Pi_{\Gamma^\tau}(\mathcal{H}^\tau) = S^\tau$, we also have

$$f_\tau(\sigma\tau) = a_\tau, \quad f'_\tau(\sigma\tau) = -h_\tau(\sigma\tau) \delta_\tau^a,$$

where $a_\tau = (a_{\tau, 1}, \dots, a_{\tau, n})$ and $\delta_\tau^a = (\frac{\delta_{\tau, 1}}{a_{\tau, 1}}, \dots, \frac{\delta_{\tau, n}}{a_{\tau, n}})$. Keeping these two formulas in mind and using (4.24), we easily calculate that the quantity in the left-hand side of (3.2) is equal to $-\frac{1}{h_\tau^2(\sigma\tau)}$. This quantity must, therefore, be negative.

Assume now that (3.2) holds. Let us prove the existence of $\kappa \in (0, 1)$ and a metric \mathcal{H}^τ on \mathcal{X}_κ^τ such that $\text{Ric}(\mathcal{H}^\tau) = T$ on \mathcal{X}_κ^τ , $\mathcal{H}_{\Gamma^\tau}^\tau = R^\tau$, and $\Pi_{\Gamma^\tau}(\mathcal{H}^\tau) = S^\tau$. Our strategy will be quite similar to the strategy we chose to handle Theorem 3.4. Consider the system of ordinary differential equations

$$\begin{aligned} f_\tau''(r) &= \tilde{F}(h_\tau(r), f_\tau(r), f_\tau'(r), \phi(r), \phi'(r)), \\ h_\tau'(r) &= K(h_\tau(r), f_\tau(r), f_\tau'(r), \phi'(r)), \end{aligned} \quad (4.25)$$

for the unknown functions f_τ and h_τ . We supplement this system with the conditions

$$\begin{aligned} f_\tau(\sigma\tau) &= a_\tau, \\ f_\tau'(\sigma\tau) &= -(H_2(a_\tau, \phi(\sigma\tau)) + (1 - H_1(a_\tau, \delta_\tau^a)))^{-\frac{1}{2}} \delta_\tau^a, \\ h_\tau(\sigma\tau) &= (H_2(a_\tau, \phi(\sigma\tau)) + (1 - H_1(a_\tau, \delta_\tau^a)))^{-\frac{1}{2}}. \end{aligned} \quad (4.26)$$

Note that, thanks to (3.2), the right-hand sides of the last two formulas are well-defined. The standard theory of ordinary differential equations (specifically, the Picard-Lindelöf theorem) tells us that problem (4.25)–(4.26) has a solution. To be more precise, for some number $\kappa \in (0, 1)$, there exist smooth functions $f_\tau : J_\kappa^\tau \rightarrow (0, \infty)^n$ and $h_\tau : J_\kappa^\tau \rightarrow (0, \infty)$ solving (4.25) on J_κ^τ and satisfying (4.26). With these functions at hand, we define a G -invariant Riemannian metric \mathcal{H}^τ on \mathcal{X}_κ^τ by formulas (4.22) and (4.23). It then follows from (4.25) that

$$\text{Ric}(\Psi^*\mathcal{H}^\tau) = \hat{\sigma}(r) dr \otimes dr + T_r, \quad r \in J_\kappa^\tau,$$

for some $\hat{\sigma} : J_\kappa^\tau \rightarrow \mathbb{R}$. Employing Lemma 4.2 and arguing as in Section 4.2, one demonstrates that $\hat{\sigma}$ must be identically equal to 1 on J_κ^τ . This means $\text{Ric}(\mathcal{H}^\tau) = T$ on \mathcal{X}_κ^τ . Conditions (4.26) imply that $\mathcal{H}_{\Gamma^\tau}^\tau = R^\tau$ and $\Pi_{\Gamma^\tau}(\mathcal{H}^\tau) = S^\tau$.

Suppose now that statements 1 and 2 in Proposition 3.3 hold true. We may assume the metric \mathcal{H}^τ satisfies (4.22) and (4.23). Then the functions f_τ and h_τ solve (4.25)–(4.26) on J_κ^τ . Consider a G -invariant Riemannian metric $\check{\mathcal{H}}^\tau$ on \mathcal{X}_κ^τ such that $\text{Ric}(\check{\mathcal{H}}^\tau) = T$ on \mathcal{X}_κ^τ , $\check{\mathcal{H}}_{\Gamma^\tau}^\tau = R^\tau$, and $\Pi_{\Gamma^\tau}(\check{\mathcal{H}}^\tau) = S^\tau$. Our objective is to show that $\check{\mathcal{H}}^\tau$ coincides with \mathcal{H}^τ . By analogy with (4.22), we write

$$\Psi^*\check{\mathcal{H}}^\tau = \check{h}_\tau^2(r) dr \otimes dr + \check{\mathcal{H}}_r^\tau, \quad r \in J_\kappa^\tau.$$

In the right-hand side, $\check{h}_\tau : J_\kappa^\tau \rightarrow (0, \infty)$ is a smooth function. The tensor field $\check{\mathcal{H}}_r^\tau$ is a G -invariant Riemannian metric on G/K . There are smooth functions $\check{f}_{\tau,1}, \dots, \check{f}_{\tau,n}$ from J_κ^τ to $(0, \infty)$ such that

$$\check{\mathcal{H}}_r^\tau(X, Y) = \check{f}_{\tau,1}^2(r) Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \dots + \check{f}_{\tau,n}^2(r) Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \quad X, Y \in \mathfrak{p}.$$

It will be convenient for us to denote $\check{f}_\tau = (\check{f}_{\tau,1}, \dots, \check{f}_{\tau,n})$. Because $\text{Ric}(\check{\mathcal{H}}^\tau) = T$, $\check{\mathcal{H}}_{\Gamma^\tau}^\tau = R^\tau$, and $\Pi_{\Gamma^\tau}(\check{\mathcal{H}}^\tau) = S^\tau$, formulas (4.25)–(4.26) would still hold on J_κ^τ if we substituted \check{f}_τ , \check{f}_τ' , \check{f}_τ'' , \check{h}_τ , and \check{h}_τ' in them for f_τ , f_τ' , f_τ'' , h_τ , and h_τ' . The standard theory of ordinary differential equations then implies that $\check{f}_\tau = f_\tau$ and $\check{h}_\tau = h_\tau$ on J_κ^τ . Consequently, $\check{\mathcal{H}}^\tau$ coincides with \mathcal{H}^τ . Thus, the proof of Proposition 3.3 is complete. One may also establish this proposition by adapting the methods developed in the paper [26]. Such an approach requires a little more work but avoids using Lemma 4.2.

4.5 Proof of Theorems 3.1 and 3.2

Choose $\tau \in [0, 1]$ and $\beta > 0$. Let R' and S' be the symmetric G -invariant (0,2)-tensor fields on Γ^τ satisfying the formulas

$$\begin{aligned} ((\Psi(\tau, \cdot))^* R')(X, Y) &= ((1 - \tau)a_1 + \tau b_1) Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \dots + ((1 - \tau)a_n + \tau b_n) Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \\ ((\Psi(\tau, \cdot))^* S')(X, Y) &= \beta Q(X, Y), \quad X, Y \in \mathfrak{p}. \end{aligned}$$

Assuming β is sufficiently large and using Proposition 3.3, we obtain a neighbourhood \mathcal{X}^τ of Γ^τ and a Riemannian metric \mathcal{G}^τ on M such that $\text{Ric}(\mathcal{G}^\tau) = T$ on \mathcal{X}^τ , $\mathcal{G}_{\Gamma^\tau}^\tau = R'$, and $\Pi_{\Gamma^\tau}(\mathcal{G}^\tau) = S'$. Theorems 3.1 and 3.2 immediately follow.

5 Variants and generalizations

This section aims to explain how some of the assumptions imposed in Sections 2 and 3 may be modified, relaxed, or even removed. In particular, Propositions 5.1 and 5.2 appearing below are more general versions of Theorem 3.4.

5.1 The orbit space M_1/G need not be homeomorphic to \mathbb{R}

We stipulated in the beginning of Section 2 that the orbit space M_1/G must be one-dimensional. Consequently, M_1/G has to be homeomorphic to the real line \mathbb{R} , the closed interval $[-1, 1]$, the half-line $[1, \infty)$, or the circle \mathcal{S}^1 . Sections 2, 3, and 4 discussed the first of these four possibilities. It is worth clarifying that one can apply Theorems 3.1, 3.2, and 3.4, as well as Proposition 3.3, in the other three situations. Suppose, for instance, that M_1/G is homeomorphic to $[-1, 1]$. Let M_0 be the set of those points in M_1 that lie on principal G -orbits. Choose a diffeomorphism

$$\Phi_0 : (0, 1) \times (G/K) \rightarrow M_0$$

such that the map $\Phi_0(s, \cdot)$ is G -equivariant for every $s \in (0, 1)$. Given two numbers $\sigma' < \sigma''$ from $(0, 1)$, define

$$M = \Phi_0([\sigma', \sigma''] \times G/K).$$

We introduce a new manifold

$$\tilde{M}_1 = \Phi_0\left(\left(\frac{\sigma'}{2}, \frac{1 + \sigma''}{2}\right) \times G/K\right).$$

The group G acts naturally on \tilde{M}_1 . The orbit space \tilde{M}_1/G is homeomorphic to \mathbb{R} . Obviously, M is a subset of \tilde{M}_1 . If we replace M_1 with \tilde{M}_1 , we will find ourselves in the situation described in Section 2. It will then be possible to apply Theorems 3.1, 3.2, and 3.4, as well as Proposition 3.3, to study the solvability of the prescribed Ricci curvature equation on M . Analogous reasoning works in the cases where M_1/G is homeomorphic to $[1, \infty)$ and \mathcal{S}^1 .

5.2 The space G/K may be an abelian Lie group

The arguments in Sections 3 and 4 rely on Hypothesis 2.1. Assume that this hypothesis is not satisfied. Instead, suppose G/K is an abelian Lie group. The K -module \mathfrak{p} can then be written in the form (2.3) with the K -modules \mathfrak{p}_k being one-dimensional for all $k = 1, \dots, n$. It is possible to adapt the theorems of Section 3 to this setting. Let us outline the required changes. As before, one can construct the diffeomorphism Ψ and write down formula (2.4). In our current situation, however, it is not necessarily the case that there are smooth functions $\hat{\phi}_1, \dots, \hat{\phi}_n$ from $[0, 1]$ to $(0, \infty)$ obeying equality (2.5). Assume that such functions do exist. Suppose also that one can find positive numbers a_1, \dots, a_n and b_1, \dots, b_n such that (2.6) holds. Thus, we demand that T and R be diagonal with respect to (2.3). It is then possible to prove the assertions of Theorems 3.1, 3.2, and 3.4 using the reasoning of Section 4. Moreover, if G/K is an abelian Lie group, the constants $(\beta_k)_{k=1}^n$ and $(\gamma_{k,l}^m)_{k,l,m=1}^n$ are all equal to 0. This means we can choose an arbitrary $\bar{\rho} > 0$ and define the function ρ_0 in Theorem 3.4 by setting $\rho_0(p, q) = \bar{\rho}$ for $p, q \in (0, \infty)^2$. Note that σ_0 will depend on the choice of $\bar{\rho}$.

A word of warning: While it is easy to show that statement 2 of Proposition 3.3 implies statement 1 in our current setting, establishing the converse implication may be problematic. Roughly speaking, this is because, when Hypothesis 2.1 does not hold, the metric \mathcal{H}^τ need not be diagonal with respect to (2.3). For the same reason, proving the assertion of Proposition 3.3 that concerns \mathcal{H}^τ may be troublesome with our methods.

5.3 The tensor field T need not be positive-definite

We assumed in Section 2 that the tensor field T was positive-definite. This assumption enabled us to construct the diffeomorphism Ψ satisfying formulas (2.4)–(2.5). It also implied that $0 < \hat{\phi}_i(t)$ for all $i = 1, \dots, n$

and $t \in [0, 1]$. We can replace T by a tensor field that is not necessarily positive-definite (but merely nondegenerate in a direction transverse to the G -orbits) and still obtain variants of Theorems 3.1, 3.2, and 3.4. A few additional requirements will have to be imposed. Let us explain this in more detail. Assume Hypothesis 2.1 holds. Consider a symmetric G -invariant $(0, 2)$ -tensor field \tilde{T} on M . Suppose there exists a diffeomorphism

$$\tilde{\Psi} : [0, 1] \times (G/K) \rightarrow M$$

such that $\tilde{\Psi}(t, \cdot)$ is G -equivariant whenever $t \in [0, 1]$ and the equality

$$\tilde{\Psi}^* \tilde{T} = \tilde{\sigma}^2 dt \otimes dt + \tilde{T}_t \quad (5.1)$$

holds true. In the right-hand side, $\tilde{\sigma}$ must be a positive number. Accordingly, \tilde{T}_t has to be a G -invariant $(0, 2)$ -tensor field on G/K for every $t \in [0, 1]$. It is clear that the formula

$$\tilde{T}_t(X, Y) = \tilde{\phi}_1(t) Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \cdots + \tilde{\phi}_n(t) Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \quad X, Y \in \mathfrak{p},$$

is then satisfied for some smooth functions $\tilde{\phi}_1, \dots, \tilde{\phi}_n$ from $[0, 1]$ to \mathbb{R} . An analysis of the reasoning in Sections 4.4 and 4.5 leads to the following conclusion: The assertions of Theorems 3.1 and 3.2 would still hold if we replaced T in them by \tilde{T} . The situation with Proposition 3.3 is more complicated. Roughly speaking, when $\tilde{\Psi}$ and \tilde{T} appear in lieu of Ψ and T throughout Section 3.1, statement 1 of this proposition is equivalent to statement 2. Yet our methods do not yield the assertion about $\tilde{\mathcal{H}}^\tau$.

Fix $\tilde{\alpha} > 0$ obeying the inequality

$$|\tilde{\phi}_i(t)| \leq \tilde{\alpha}, \quad i = 1, \dots, n, \quad t \in [0, 1].$$

Assume that

$$\Gamma^0 = \tilde{\Psi}(\{0\} \times G/K), \quad \Gamma^1 = \tilde{\Psi}(\{1\} \times G/K).$$

Accordingly, we have

$$\begin{aligned} ((\tilde{\Psi}(0, \cdot))^* R^0)(X, Y) &= a_1^2 Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \cdots + a_n^2 Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \\ ((\tilde{\Psi}(1, \cdot))^* R^1)(X, Y) &= b_1^2 Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \cdots + b_n^2 Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \quad X, Y \in \mathfrak{p}. \end{aligned}$$

Recall that (3.4) holds true. We will now present a variant of Theorem 3.4 with T replaced by \tilde{T} .

Proposition 5.1. *There exist functions $\tilde{\rho}_0 : (0, \infty)^2 \rightarrow (0, \infty)$ and $\tilde{\sigma}_0 : (0, \infty)^3 \rightarrow (0, \infty)$ satisfying the following statement: if the formulas*

$$\sum_{i=1}^n d_i \left(\frac{\max \{ \tilde{\phi}_i(t), 0 \}}{\omega_2^2} + \frac{\min \{ \tilde{\phi}_i(t), 0 \}}{\omega_1^2} \right) > \tilde{\rho}_0(\omega_1, \omega_2), \quad \tilde{\sigma} < \tilde{\sigma}_0(\tilde{\alpha}, \omega_1, \omega_2), \quad t \in [0, 1],$$

and the formulas

$$|a_i - b_i| \leq \tilde{\sigma}^2, \quad \left| \frac{d}{dt} \tilde{\phi}_i(t) \right| \leq \tilde{\sigma}^2, \quad i = 1, \dots, n, \quad t \in [0, 1],$$

hold true, the manifold M supports a G -invariant Riemannian metric \mathcal{G} such that $\text{Ric}(\mathcal{G}) = \tilde{T}$ and $\mathcal{G}_{\partial M} = R$.

To carry out the proof, one has to repeat the arguments in Sections 4.2 and 4.3 with minor modifications. We will not discuss this further. It is worth clarifying, however, that one can choose the functions $\tilde{\rho}_0$ and $\tilde{\sigma}_0$ in the proposition above to be independent of \tilde{T} and R .

The reasoning in Section 5.3 persists when Hypothesis 2.1 is replaced by assumptions on G/K , \tilde{T} , and R similar to those described in Section 5.2. In particular, Proposition 5.1 holds if two requirements are met: First, G/K is an abelian Lie group. Second, \tilde{T} and R have appropriate diagonal structure with respect to (2.3).

Instead of assuming the existence of $\tilde{\Psi}$ above, one may assume there is a diffeomorphism $\check{\Psi}$ such that (5.1) holds with $\check{\Psi}$ substituted for $\tilde{\Psi}$ and $-\tilde{\sigma}^2$ substituted for $\tilde{\sigma}^2$. The techniques in the present paper seem to be effective for treating this case. We will not dwell on any further details.

5.4 One more generalization

Our next result is, again, a variant of Theorem 3.4. It shows that one can replace inequalities (3.6) with less restrictive inequalities at the expense of changing the function σ_0 . Assume Hypothesis 2.1 holds true. In the beginning of Section 3.2, we fixed a number α satisfying (3.3) and a pair ω_1, ω_2 obeying (3.4). Let us also choose $c_1, c_2 > 0$ such that

$$|a_i - b_i| \leq c_1 \sigma^2, \quad \left| \frac{d}{dt} \hat{\phi}_i(t) \right| \leq c_2 \sigma^2, \quad i = 1, \dots, n, \quad t \in [0, 1].$$

We will now formulate our next result and make a few comments.

Proposition 5.2. *There exist functions $\check{\rho}_0 : (0, \infty)^2 \rightarrow (0, \infty)$ and $\check{\sigma}_0 : (0, \infty)^5 \rightarrow (0, \infty)$ that satisfy the following assertion: if the formulas*

$$\sum_{i=1}^n d_i \hat{\phi}_i(t) > \check{\rho}_0(\omega_1, \omega_2), \quad \sigma < \check{\sigma}_0(\alpha, \omega_1, \omega_2, c_1, c_2), \quad t \in [0, 1],$$

hold, then M carries a G -invariant Riemannian metric \mathcal{G} with $\text{Ric}(\mathcal{G}) = T$ and $\mathcal{G}_{\partial M} = R$.

One can choose $\check{\rho}_0$ and $\check{\sigma}_0$ here to be independent of T and R .

Proposition 5.2 is more general than Theorem 3.4. However, the intuition behind it seems harder to grasp. In the situation where $c_1 = c_2 = 1$, the two results are equivalent. To prove Proposition 5.2, it suffices to follow the reasoning from Sections 4.2 and 4.3. Only small changes to the arguments are needed. We leave it up to the reader to work out the details.

Let us point out that Proposition 5.2 would still hold if, instead of Hypothesis 2.1, one imposed assumptions on G/K , T , and R similar to those in Section 5.2. Roughly speaking, this means G/K may be an abelian Lie group as long as T and R are diagonal. Also, we can obtain an analogue of Proposition 5.2 in the case where T is replaced with a tensor field that is not necessarily positive-definite. In order to do so, we simply have to repeat the reasoning from Section 5.3.

6 An example

The constructions discussed above are rather abstract. It seems appropriate to provide a specific example. We will show that the results of Section 3 can be used to investigate problem (2.1)–(2.2) on a solid torus with a neighbourhood of the core circle removed. The reader will find related material in [26].

Let us introduce some notation. For $\epsilon > 0$, define

$$\begin{aligned} \mathcal{T}_\epsilon &= \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq \epsilon^2\} \times \{(w, z) \in \mathbb{R}^2 \mid w^2 + z^2 = 1\}, \\ \mathcal{T}_\epsilon^\circ &= \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < \epsilon^2\} \times \{(w, z) \in \mathbb{R}^2 \mid w^2 + z^2 = 1\}, \\ \mathcal{T}_0 &= \{(0, 0)\} \times \{(w, z) \in \mathbb{R}^2 \mid w^2 + z^2 = 1\}. \end{aligned}$$

It is obvious that \mathcal{T}_ϵ is a solid torus embedded into $\mathbb{R}^2 \times \mathbb{R}^2$. The set $\mathcal{T}_\epsilon^\circ$ is the interior of \mathcal{T}_ϵ , and \mathcal{T}_0 is the core circle of \mathcal{T}_ϵ . Given $\chi \in (0, 1)$, one easily sees that \mathcal{T}_χ° is a neighbourhood of \mathcal{T}_0 in \mathcal{T}_1 . We will now demonstrate that Theorems 3.1, 3.2, and 3.4 can help study problem (2.1)–(2.2) on $\mathcal{T}_1 \setminus \mathcal{T}_\chi^\circ$. Employing these theorems, one can partially recover the results of [26] and produce new results as well.

Assume G is equal to the product $\text{SO}(2) \times \text{SO}(2)$, where $\text{SO}(2)$ is the special orthogonal group of \mathbb{R}^2 . Define M_1 to be the difference $\mathcal{T}_2^\circ \setminus \mathcal{T}_{\frac{\chi}{2}}$. The standard action of $\text{SO}(2)$ on \mathbb{R}^2 gives rise to an action of G on M_1 . The orbits of this action are the tori \mathcal{T}_ϵ with $\epsilon \in (\frac{\chi}{2}, 2)$. We suppose $o = ((1, 0), (1, 0)) \in M_1$ and choose a smooth bijective function $\lambda : \mathbb{R} \rightarrow (\frac{\chi}{2}, 2)$ with positive derivative. The isotropy group of o consists of nothing but the identity element in G . The map $\Phi : \mathbb{R} \times G \rightarrow M_1$ given by the formula

$$\Phi \left(s, \begin{pmatrix} \cos p & -\sin p \\ \sin p & \cos p \end{pmatrix}, \begin{pmatrix} \cos q & -\sin q \\ \sin q & \cos q \end{pmatrix} \right) = ((\lambda(s) \cos p, \lambda(s) \sin p), (\cos q, \sin q)),$$

$p, q \in [0, 2\pi),$

is a diffeomorphism. Setting $\sigma' = \lambda^{-1}(\chi)$ and $\sigma'' = \lambda^{-1}(1)$, we obtain

$$M = \Phi([\sigma', \sigma''] \times G) = \mathcal{T}_1 \setminus \mathcal{T}_\chi^\circ.$$

The components Γ^0 and Γ^1 of the boundary ∂M are equal to the boundaries $\partial \mathcal{T}_\chi$ and $\partial \mathcal{T}_1$, respectively.

Consider a symmetric $(0, 2)$ -tensor field T on M . It is convenient for us to assume that T is positive-definite although, as explained in Section 5.3, this assumption can be relaxed. Suppose T is rotationally symmetric in the sense of [11, 26]. This means T is G -invariant and diagonal with respect to the cylindrical coordinates on \mathcal{T}_1 . Consider also a symmetric positive-definite $(0, 2)$ -tensor field R on ∂M . We need to impose a restriction on the form of R as well. Namely, we suppose R is G -invariant and diagonal in the coordinates induced on ∂M by the cylindrical coordinates on \mathcal{T}_1 .

In the current setting, Theorems 3.1 and 3.2 (along with the remarks of Section 5.2) imply local solvability of problem (2.1)–(2.2). Similar results were obtained in [26] by different methods. Theorem 3.4 yields a sufficient condition for the solvability of (2.1)–(2.2) on all of M . No such condition previously appeared in the literature.

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